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A THREE-DIMENSIONAL INVERSE PROBLEM FOR A PHYSICALLY NON-LINEAR INHOMOGENEOUS MEDIUM†

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A three-dimensional linearly elastic (viscoelastic) domain (finite or infinite) containing a physically non-linear inclusion of arbitrary shape is considered. The possibility of creating a prescribed uniform stress–strain state in the inclusion by a suitable choice of loads on the outer boundary of the domain is considered. A solution is constructed in closed form. Some examples are considered, including, in particular, the case of an ellipsoidal inclusion with the property of non-linear creep. © 2005 Elsevier Ltd. All rights reserved.

Two-dimensional problems for finite elastic [1] and viscoelastic [2] domains containing a physically non-linear inclusion (PNI) of arbitrary shape have been investigated, in which the application of suitable external loads creates a given uniform stress–strain state (SSS) in the inclusion. The solutions constructed in [1, 2] contained, apart from quantities characterizing the SSS in the inclusion and the properties of the basic medium, only a representing function associated with the boundary of the inclusion. In this paper results obtained in [3] will be used to solve similar problems (moreover, in closed form) in three dimensions. Special attention will be given to an ellipsoidal inclusion (EPNI), when the solution can be extended to infinitely distant points, the SSS at which will also be uniform. In particular, some problems of the deformation and fracture of ellipsoidal inclusions under conditions of creep will be considered.

1. THE STRESS–STRAIN STATE OF A THREE-DIMENSIONAL DOMAIN CONTAINING A PHYSICALLY NON-LINEAR INCLUSION WITH A GIVEN STRESS–STRAIN STATE

Consider an elastic (or viscoelastic) domain υ of space with a physically non-linear inclusion υ^*. The outer and inner boundaries of the domain υ are piecewise-smooth surfaces S and S^* (the latter separates υ from υ^*).

Hooke's law holds in the basic medium υ

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn}, \quad \sigma_{kl} = b_{klmn}\varepsilon_{mn}, \quad k, l = 1, 2, 3 \tag{1.1}$$

where ε_{kl}, σ_{kl}, a_{klmn} and b_{klmn} are the components of the strain, stress, elastic compliance, and elastic moduli tensors; repeated indices indicate summation from 1 to 3. The system of coordinates $Ox_1x_2x_3$ is chosen so that $(0, 0, 0) \in \upsilon^*$. If the quantities a_{klmn} and b_{klmn} $(k, l, m, n = 1, 2, 3)$ are understood as the corresponding Volterra operators [2], then equalities (1.1) will be the constitutive equations for a linear viscoelastic medium.

For the inclusion υ^* we have [1, 2]

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$$\varepsilon_{kl}^* = F_{kl}(\sigma_{mn}^*), \quad \sigma_{kl}^* = G_{kl}(\varepsilon_{mn}^*), \quad k, l, m, n = 1, 2, 3 \tag{1.2}$$

where F_{kl} and G_{kl} are the components of mutually inverse non-linear tensor operators.

The formulation of the main problem is analogous to that of the problems considered in [1, 2]: what displacements u_{k0} must be communicated to the boundary S (or loads p_{k0} applied to S) so as to create in the inclusion υ^* the required uniform (i.e. independent of the coordinates x_k) stress–strain state (SSS), characterized by stresses $\sigma_{kl}^* = \sigma_{kl}^*(t)$ and strains $\varepsilon_{kl}^* = \varepsilon_{kl}^*(t)$ (k, l = 1, 2, 3; t is the time or a loading parameter) satisfying Eqs (1.2)? At the initial time $t = 0$ the domain $\upsilon^* \cup \upsilon$ was in an unstrained state. At the boundary S^* the fields of loads $p_k = \sigma_{kl}n_l$ (where n_l are the components of a unit vector normal to S) and displacements u_k (k = 1, 2, 3) are continuous. The problem is geometrically linear.

Since the strains ε_{kl}^* and υ^* are independent of the coordinates, the displacement vector $\mathbf{u}^*$ will be a linear function of x_k, that is, if it is assumed that $\mathbf{u}^* = 0$ at the point $(0, 0, 0) \in \upsilon^*$, then

$$u_k^* = (\omega_{kl}^* + \varepsilon_{kl}^*)x_l \quad (k = 1, 2, 3) \tag{1.3}$$

where ω_{kl}^* are the components of an antisymmetric tensor defining a uniform rotation vector in υ^*, which is also assumed to be given (for example, $\omega_{kl}^* = 0$; $k, l = 1, 2, 3$).

In [3] an elastic space was considered with a physically non-linear inclusion subjected at infinity to the action of external forces, corresponding to which in the uniform elastic medium (that is, when there is no inclusion) there were fields of stresses $\sigma_{kl}^\infty = \sigma_{kl}^\infty(\mathbf{r})$ and displacements $u_k^\infty = u_k^\infty(\mathbf{r})$; the following relations were obtained

$$u_k(\mathbf{r}) = u_k^\infty(\mathbf{r}) + F_k(\mathbf{r}), \quad F_k(\mathbf{r}) = \int\limits_{\upsilon^*} \Phi_{pq}(\boldsymbol{\xi}) U_{kp,q}(\mathbf{r} - \boldsymbol{\xi}) d\upsilon(\boldsymbol{\xi})$$

$$k = 1, 2, 3; \quad \mathbf{r} = (x_1, x_2, x_3), \quad 0 \le |\mathbf{r}| < \infty, \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \upsilon^* \tag{1.4}$$

$$\Phi_{pq} = \sigma_{pq}^* - b_{pqmn}\varepsilon_{mn}^*, \quad 2\varepsilon_{mn}^* = u_{m,n}^* + u_{n,m}^*$$

where U_{kp} are the components of Green's tensor and the subscript q after a comma denotes a derivative with respect to x_q; the other quantities are defined by formulae (1.1)–(1.3). (Note that the minus sign in front of the integral in formula (1.4) in [3] is an error, as is obvious from the formulae presented there on p. 13.) In what follows we shall assume that the components of the displacements, stresses, and strains occurring in (1.4) also depend on t.

In the direct problem, that is, given the functions $u_k^\infty = u_k^\infty(\mathbf{r})$, Eqs (1.4) are non-linear integral equations for $u_k = u_k(\mathbf{r})$, $0 \le |\mathbf{r}| \le \infty$. It has been shown [3] that the solution of these equations (in case it exists), that is, the vector of displacements and the field of stresses corresponding to it by Eqs (1.1) and (1.2), satisfy the previously mentioned continuity conditions at the surface S^* and that the equilibrium conditions are satisfied throughout the space.

In the case considered here, that of the inverse problem, where, according to Eqs (1.3), the given quantities are $u_k^* = u_k^*(\mathbf{r})$, $\mathbf{r} \in \upsilon^*$, while the quantities Φ_{pq} occurring in formulae (1.4) are independent of ξ, one can determine the required functions $u_{k0} = u_{k0}(\mathbf{r}_s)$, $\mathbf{r}_s \in S$, that is, the vector of displacements on the outer boundary of the domain υ, from relations (1.4). Indeed, by (1.4), we obtain the following equalities for the components u_k^∞ in the physically non-linear inclusion

$$u_k^\infty(\mathbf{r}) = u_k^*(\mathbf{r}) - F_k(\mathbf{r}), \quad k = 1, 2, 3; \quad \mathbf{r} \in \upsilon^* \tag{1.5}$$

The functions $u_k^\infty(\mathbf{r})$ are analytic in the domain υ^* (at least, in the case of an isotropic elastic domain υ, when U_{kp} are the components of the Kelvin–Somigliana tensor and are expressed in terms of known harmonic and biharmonic potentials [4]). They may therefore be continued into the domain υ and beyond (if υ is finite), i.e. beyond the boundary S. In the case when the domain υ is infinite and u_k^∞ are polynomials of degree $n < \infty$, such a continuation is also possible, that is, the field of the stress tensor at infinity is a polynomial of degree $n - 1$. For example, in the case of an ellipsoidal physically non-linear inclusion $n = 1$, that is, the stresses σ_{kl}^∞ are finite as $|\mathbf{r}| \to \infty$ [3].

For known functions $u_k^\infty = u_k^\infty(\mathbf{r})$, $0 \le |\mathbf{r}| < \infty$, we find from relations (1.4) that $u_k = u_k(\mathbf{r})$, $\mathbf{r} \in \upsilon$ and $u_{k0} = u_{k0}(\mathbf{r}_s)$ (or $p_{k0} = p_{k0}(\mathbf{r}_s)$), $\mathbf{r}_s \in S$ (k = 1, 2, 3).

Note that, as in the two-dimensional problem [1, 2], the SSS in υ is uniquely defined by the geometry of the inclusion υ^*, the quantities σ_{kl}^* and ε_{kl}^*, and also the elastic (viscoelastic) characteristics of υ, while its outer boundary S does not affect the SSS. The loads p_{k0} on the boundary S are determined by its shape and the already determined SSS in υ.

The solution of the inverse problem considered above is unique, that is, for a known field u_k^* in the domain υ^*, the SSS in the domain υ and the corresponding displacements and loads on the boundary S are uniquely defined [1, 2]. The converse is also true: Given u_k or p_k on the boundary S and certain restrictions imposed on relations (1.2) (which reduce to the assumption that the process of deformation of the material of the inclusion is stable [1, 2]), the SSS in the domain $\upsilon^* \cup \upsilon$ is uniquely defined, that is, the previously found functions $u_{k0} = u_{k0}(\mathbf{r}_s)$ (or $p_{k0} = p_{k0}(\mathbf{r}_s)$), $\mathbf{r}_s \in S$, determine a uniform SSS in the domain υ^*. The proof duplicates that of [1, 2], except that the two-dimensional integrals are replaced by the analogous three-dimensional ones.

2. EXAMPLES

Let us consider the case of an ellipsoidal inclusion with semi-axes a_k ($k = 1, 2, 3$), for which the functions $F_k(\mathbf{r})$ in relations (1.4) are linear when $\Phi_{pq} = \text{const}$ [3, 4]. Therefore, in view of relations (1.3) and (1.5), $u_k^\infty = u_k^\infty(\mathbf{r})$ will also be linear. Consequently, as already pointed out, as $|\mathbf{r}| \to \infty$ the stresses σ_{kl}^∞ will be finite.

It is not hard to establish the relation between the SSS in the ellipsoidal inclusion and at infinity. Indeed, Eshelby, in his classical paper [4] for the case of an inclusion subject to a transformation accompanied by a free uniform deformation ε_{kl}^T, obtained relations of the type (1.4) for $u_k^\infty = 0$ and $\Phi_{pq} = -b_{pqmn}\varepsilon_{mn}^T$, from which we obtain the following relations between the free and constrained deformations:

$$\begin{aligned}&\varepsilon_{kl}^* = S_{klmn}\varepsilon_{mn}^T, \quad 2S_{klmn} = -b_{pqmn}\int_{\upsilon^*}[U_{kp,ql}(\mathbf{r}-\boldsymbol{\xi}) + \\ &+ U_{lp,qk}(\mathbf{r}-\boldsymbol{\xi})]d\upsilon(\boldsymbol{\xi}), \quad k, l, m, n = 1, 2, 3; \quad \mathbf{r}, \boldsymbol{\xi} \in \upsilon^*\end{aligned} \tag{2.1}$$

The components S_{klmn} of Eshelby's tensor $\mathbf{S}$ [4] are independent of x_k ($k = 1, 2, 3$). It then follows from relations (1.1)–(1.4) and (2.1) for the case of an ellipsoidal inclusion that

$$\begin{aligned}&u_k^\infty = (\omega_{kl}^\infty + \varepsilon_{kl}^\infty)x_l, \quad \varepsilon_{kl}^\infty = \varepsilon_{kl}^* + S_{klmn}(\tilde{\varepsilon}_{mn}^* - \varepsilon_{mn}^*)\\ &\varepsilon_{kl}^\infty = a_{klmn}\sigma_{mn}^\infty, \quad \tilde{\varepsilon}_{kl}^* = a_{klmn}\sigma_{mn}^*, \quad \omega_{kl}^\infty = \omega_{kl}^* + \Pi_{klmn}(\varepsilon_{mn}^* - \tilde{\varepsilon}_{mn}^*)\\ &2\Pi_{klmn} = b_{pqmn}\int_{\upsilon^*}[U_{kp,kl}(\mathbf{r}-\boldsymbol{\xi}) - U_{lp,qk}(\mathbf{r}-\boldsymbol{\xi})]d\upsilon(\boldsymbol{\xi})\end{aligned} \tag{2.2}$$

For an isotropic elastic medium υ, the components of the tensors $\mathbf{S}$ and $\mathbf{\Pi}$ in (2.1) and (2.2) are defined as follows [4] (where ν is Poisson's ratio):

$$\begin{aligned}&S_{kkkk} = Qa_k^2 I_{kk} + RI_k, \quad S_{kkll} = Qa_l^2 I_{kl} - RI_k\\ &2S_{klkl} = 2S_{kllk} = Q(a_k^2 + a_l^2)I_{kl} + R(I_k + I_l); \quad \Pi_{klkl} = -\Pi_{kllk} = \frac{I_l - I_k}{8\pi}\\ &Q = \frac{3}{8\pi(1-\nu)}, \; R = \frac{1-2\nu}{8\pi(1-\nu)}; \; I_k = 2\pi a_1 a_2 a_3\int_0^\infty \frac{du}{(a_k^2+u)\Delta}, \; I_{kk} = 2\pi a_1 a_2 a_3\int_0^\infty \frac{du}{(a_k^2+u)^2\Delta}\\ &3I_{kl} = 2\pi a_1 a_2 a_3\int_0^\infty \frac{du}{(a_k^2+u)(a_l^2+u)\Delta}; \quad \Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)\end{aligned} \tag{2.3}$$

($k, l = 1, 2, 3, k \neq l$; there is no summation over k and l); the remaining components vanish: $S_{klmn} = 0$ and $\Pi_{klmn} = 0$.

The quantities I_k, I_{kk} and I_{kl} are expressed in terms of elliptic integrals of the first and second kind and may be determined if any two of the I_k are known, as, for example, in the case of oblate and prolate spheroids, when I_k are elementary functions of a_1, a_2 and a_3 and the following equalities hold:

if $a_1 = a_2 = a, a_3 = \delta a, \delta < 1$

$$I_1 = I_2 = 2\pi\delta(1-\delta^2)^{-3/2}[\arccos\delta - \delta(1-\delta^2)^{1/2}], \quad I_3 = 4\pi - 2I_1$$

$$I_{11} = I_{22} = 3I_{12} = \frac{3I_1 - 4\pi\delta^2}{4a^2(1-\delta^2)}, \quad I_{13} = I_{23} = \frac{4\pi - 3I_1}{3a^2(1-\delta^2)} \tag{2.4}$$

$$I_{33} = \frac{4\pi(1-3\delta^2) + 6I_1\delta^2}{3a^2\delta^2(1-\delta^2)}$$

if $a_1 = a, a_2 = a_3 = \delta a, \delta < 1$

$$I_1 = 4\pi - 2I_2, \; I_2 = I_3 = \frac{2\pi}{\delta}\left(\frac{1}{\delta^2} - 1\right)^{-3/2}\left[\frac{1}{\delta}\left(\frac{1}{\delta^2} - 1\right)^{1/2} - \operatorname{arch}\frac{1}{\delta}\right]$$

$$I_{11} = \frac{4\pi(3-\delta^2) - 6I_2}{3a^2(1-\delta^2)}, \quad I_{22} = I_{33} = 3I_{23} = \frac{4\pi - 3I_2\delta^2}{4a^2\delta^2(1-\delta^2)}, \; I_{12} = I_{13} = \frac{3I_2 - 4\pi}{3a^2(1-\delta^2)} \tag{2.5}$$

We also note that in the case of an elliptical cylinder, when $a_3 \to \infty$ and the following equalities hold [4]

$$I_1 = \frac{4\pi a_2}{a_1 + a_2}, \quad I_2 = \frac{4\pi a_1}{a_1 + a_2}, \; I_3 = 0, \; I_{12} = \frac{4\pi}{3(a_1 + a_2)^2}$$

$$I_{kk} = \frac{4\pi}{3a_k^2} - I_{12}, \quad k = 1, 2; \quad I_{k3} = 0, \quad k = 1, 2, 3$$

relations (2.2) are identical with those obtained previously in two dimensions ([1], formulae (3.4) for $\kappa = 3 - 4\nu$, corresponding to two-dimensional deformation).

As examples of the application of relations (2.2)–(2.5), we will briefly list a few problems for the case of an elastic or viscoelastic medium with an ellipsoidal inclusion, whose deformations are combinations of elastic deformations and creep deformations ε_{kl}^{*c}, that is, the constitutive equations (1.2) have the form [2]

$$\varepsilon_{kl}^* = a_{klmn}^*\sigma_{mn}^* + \varepsilon_{kl}^{*c}, \quad \dot{\varepsilon}_{mn}^{*c} = B_1 s^n (1-\omega)^{-m}\partial s/\partial\sigma_{kl}^*, \quad k, l = 1, 2, 3$$
$$\dot{\omega} = B_2 s^p (1-\omega)^{-m} \tag{2.6}$$

where $s = s(\sigma_{kl}^*)$ is a homogeneous convex function of the first degree, $\omega (0 \le \omega \le 1)$ is a damage parameter, and B_1, B_2, m, n and p are positive constants.

The inverse problems are analogous to those considered in [2] for a finite two-dimensional domain with an ellipsoidal inclusion and are formulated as follows.

Problem 1. It is required to choose stresses

$$\sigma_{kl}^\infty = \sigma_{kl}^\infty(t), \quad 0 \le t \le t_0 \tag{2.7}$$

for which the creep deformations ε_{kl}^{*c} in the domain υ^* at $t = t_0$ take prescribed values ε_{kl**}^{*c} for the least value of the damage parameter ω. The duration t_0 of the process and the stresses in υ^* are bounded:

$$0 < t_0 \le t_{**}, \quad \max_{0 \le t \le t_0} s(t) \le s_{**}$$

where t_{**} and s_{**} are given quantities.

Problem 2. Under the same restrictions due to external forces, it is required to fracture the domain υ^* at a time $t_0 \le t_{**}$ at a minimum level of energy dissipated in creep.

Problem 3a and 3b. Find stresses (2.7) for which the fracture of the ellipsoidal inclusion occurs at the required deformations ε_{kl**}^{*c}:

(a) at the least value t_0 of the duration of the application of the external force;

(b) at the least value of the dissipated energy.

In all these problems it is assumed that $\omega = 0$ and $\varepsilon_{kl}^{*c} = 0$ $(k, l = 1, 2, 3)$ for $t < 0$.

In other words, it is required to create in the inclusion an optimal (in the appropriate sense) SSS, which will be uniform. Such optimal modes of deformation and fracture of an inclusion were studied, for each of the problems listed, in [2]. Given σ_{kl}^* and ε_{kl}^* satisfying system (2.6), the stresses σ_{kl}^∞ at infinity are determined from relations (2.2) and (2.3); in particular, for spheroidal physically non-linear inclusions they are determined from relations (2.2), (2.4) and (2.5).

As before [2], Problems 1–3 may be formulated for the case of a finite elastic or viscoelastic domain υ with a physically non-linear inclusion of arbitrary shape. The appropriate optimal SSS in the inclusion must be uniform [2]. It is obtained by a suitable choice of the loads $p_{k0} = p_{k0}(t)$ on the boundary S, as determined by formulae (1.4) and (1.5).

3. SOME REMARKS ON PROBLEM (u, p)

As already pointed out [1], the inverse problem considered in Section 1 was reduced to the so-called Problem $(\mathbf{u}, \mathbf{p})$ of elasticity or viscoelasticity theory [5] for a doubly connected domain υ, on whose inner boundary S^* displacements u_k^* and loads p_k^* are given; no conditions are prescribed on the outer boundary S. Indeed, in the inclusion υ^* we know the uniform fields

$$\sigma_{kl}^* = \sigma_{kl}^*(t), \quad \varepsilon_{kl}^* = \varepsilon_{kl}^*(t), \quad \omega_{kl}^* = \omega_{kl}^*(t)$$

and so u_k^* will be determined on the boundary S^* by expressions (1.3) with $x_k \in S^*$, while $p_k^* = \sigma_{kl}^* n_l^*$, where n_k^* are the components of the outward normal to S^* (with respect to υ).

Conversely, sometimes the solution of Problem $(\mathbf{u}, \mathbf{p})$ for the domain υ may be reduced to finding the vector $\mathbf{u} = \mathbf{u}(\mathbf{r})$, $\mathbf{r} \in \upsilon$ using formulae (1.4) and (1.5). As examples of that situation, consider the following:

(a) on the boundary S^* one has $p_k^* = 0$, $u_k^* = \alpha_{kl} x_l$, $x_k \in S^*$ $(k = 1, 2, 3)$, where α_{kl} are constant quantities;

(b) $u_k^* = 0, p_k^* = \beta_{kl} n_l^*$ on S^*, $\beta_{kl} = \beta_{lk}$ are constant quantities.

These boundary conditions may be treated as follows. In case (a), we have a load-free cavity occupying the domain υ^* with displacements of the points of its boundary which are linear in x_k; in case (b) we have a rigid (non-deformable) inclusion υ^* in a uniform stressed state. We may thus assume that the inclusion is an elastic medium, whose constitutive equations (1.2) have the following respective forms

(a) $\sigma_{kl}^* = b_{klmn}^* \varepsilon_{mn}^*$ as $b_{klmn}^* \to 0$;

(b) $\varepsilon_{kl}^* = a_{klmn}^* \sigma_{mn}^*$ as $a_{klmn}^* \to 0$.

Under these assumptions, as is easily seen, $2\varepsilon_{kl}^* = \alpha_{kl} + \alpha_{lk}$ in case *a* and $\sigma_{kl}^* = \beta_{kl}$ in case (b).

Thus, relations (1.4) and (1.5) may be used with the components of the tensor $\mathbf{\Phi}$ of (1.4) defined as follows: (a) $\Phi_{kl} = -b_{klmn}\alpha_{mn}$; (b) $\Phi_{kl} = \beta_{kl}$.

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